REMARKS ON THE BOHR PHENOMENON

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Abstract. Bohr's theorem ([10]) states that analytic functions bounded by 1 in the unit disk have power series $\sum a_n z^n$ such that $\sum |a_n||z|^n < 1$ in the disk of radius 1/3 (the so-called Bohr radius.) On the other hand, it is known that there is no such Bohr phenomenon in Hardy spaces with the usual norm, although it is possible to build equivalent norms for which a Bohr phenomenon

coefficients of the power series of f, one can ask whether a version of Bohr's theorem would hold in that case. For example, given p and q, does there exist $r > 0$ such that

$$\|f\|_{H^q} \leq 1 \implies \sum_{n=0}^{\infty} |a_n|^p r^n \leq 1?$$

For $1 \leq q \leq 2$ and $p = \frac{q}{q-1}$, the above is certainly true with $r = 1$, by the

by the Hausdorff-Young theorem as noted in the Introduction. Thus, extending

Now fix $\alpha > 0$ and let $(r, n) \to (1, \infty)$ so that $(1-r)(n+1) \to \alpha$. Then

$$\liminf_{r\to 1^-}(1-r)^{1/p-1/2}C(r,p,2) \geq \frac{(2a)^{1/2}}{(ap+1)^{1/p}} \frac{\left(1-e^{-\alpha(ap+1)}\right)^{1/p}}{\left(1-r^{-2a\alpha}\right)^{1/2}}.$$

Letting $\alpha \to \infty$ we get

$$\liminf_{r\to 1^-}(1-r)^{1/p-1/2}C(r,p,2) \geq \frac{(2a)^{1/2}}{(ap+1)^{1/p}}.$$

For the choice $a = 1/(2-p)$ we have $(2a)^{1/2}/(ap+1)^{1/p} = (1-2/p)^{1/p}$, which proves (i).

To prove (iii) put $P_n(z) = z^{2n+1}V_n(z)$, where V_n is the de la Vallée-Poussin kernel, defined by $V_n = 2K_{2n+1} - K_n$, and K_n is the Fejér kernel (cf. Lemma 2.4). By Minkowski's inequality and Lemma 2.4, we have

$$\|P_n\|^s_{H^s} \leq 2\frac{2s}{2s-1}(2n+2)^{s-1} + \frac{2s}{2s-1}(n+1)^{s-1} = C_s^s(n+1)^{s-1},$$

where $C_s = (2^{s+1}s/(2s-1))^{1/s}$. Note that the k-th Fourier coefficient of V_n is 1 for $|k| \leq n$, so that

$$\frac{\|P_n\|_{p,r}}{\|P_n\|_{H^s}} \geq \frac{1}{\|P_n\|_{H^s}}\left(\left(\sum_{|k|\leq n} r^{k+2n+1}\right) + \text{a positive quantity}\right)^{1/p}$$

$$> \frac{r^{(n+1)/p}}{\|P_n\|_{H^s}}\left(\frac{1-r^{2n+1}}{1-r}\right)^{1/p} \geq \frac{r^{(n+1)/p}}{C_s(n+1)^{1-1/s}}\left(\frac{1-r^{2n+1}}{1-r}\right)^{1/p}.$$

From the above we get

$$(1-r)^{1/p+1/s-1}C(r,p,s) \geq \frac{r^{(n+1)/p}}{C_s((n+1)(1-r))^{1-1/s}}(1-r^{2n+1})^{1/p}.$$

Putting $r = 1 - 1/(n+1)$ and letting $r \to 1^-$ yields

$$\liminf_{r\to 1^-}(1-r)^{1/p+1/s-1}C(r,p,s) \geq C_s^{-1}e^{-1/p}(1-e^{-2})^{1/p} > 0.$$

It remains to prove (iv). Following [11], let $\varepsilon > 0$ and pick an integer n_ε and polyno(q822l]TJF3411.955Tf120)-398(t)098

□

Remark. It seems to be a hard task to find non-trivial estimates of $C(r, p, s)$ for $s \leq 1/2$. In fact, the "obvious" exponent might not even be the correct one (see the remark following Lemma 2.4).

It is interesting to note that if we consider functions whose first few Taylor

Proof.

This is of course a norm equivalent to the usual A^2 (the Bergman space) norm. Recall that in the usual Bergman space, there is no Bohr phenomenon (since there is none even in H^2.) Notice that if $\|f\|_X \leq 1$, then $|a_0| + \frac{|a_n|}{\sqrt{n+1}} \leq 1$, so

$$\left(\frac{|a_n|}{1 - |a_0|}\right)^{\frac{1}{n}} \leq (n+1)^{\frac{1}{2n}}.$$

The left hand side attains the value on the right for the function $f(z) =$

where T is the unit circle and $\|g\|_{BMO(T)}$ is the usual Garsia BMO norm (cf. [13].) Then for any function $g \in$ BMO, since $\|g\|_{L^1(T)} \leq \|g\|_{BMO(T)}$ (cf. [13, pp. 224-225]), if $\|f\| \leq 1$ and $n \geq 1$,

$$|a_n| \leq \|f - f(0)\|_{L^1} \leq \|f - f(0)\|_{BMO(T)} \leq 1 - |f(0)|.$$

Therefore

$$\sum_{n=0}^{\infty} |a_n| r^n \leq |f(0)| + (1 - |f(0)|)\frac{r}{1-r} \leq 1$$

whenever $r \leq \frac{1}{2}$. In fact, if X is any normed space of analytic functions such that the norm on X dominates the L^1 norm, this same argument shows that defining a new norm

$$\|f\|_{new} := |f(0)| + \|f - f(0)\|_{old}$$

will always force a positive Bohr radius of at least $\frac{1}{2}$.

It is worth noticing that Bohr's phenomenon holds for all points in the disk. Let's suppose that X is a space whose norm behaves "well" with respect to translations and dilations, namely, satisfies the following condition:

> (*) suppose f is a function in X such that $\|f\|_X \leq 1$. Then for any $z_0 \in \mathbb{D}$ and radius $r > 0$ such that the disk $D(z_0, r)$ centered at z_0 of radius r is contained in the unit disk,
>
> $$\|f(z_0 + rz)\|_X \leq 1.$$

Remark. Of course, the natural assumption that the norm in X is lower semi-continuous with respect to pointwise convergence immediately implies that if X satisfies (*), it is a subspace of H^∞, simply by letting $r \to 0$ in (*).

If such a space X satisfies the Bohr phenomenon with Bohr radius R, fix any z_0 in the disk and take the Taylor expansion of a function $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ about z_0 in the disk of radius $1 - |z_0|$. We obtain

$$\sum_{n=0}^{\infty} \left|\frac{f^{(n)}(z_0)}{n!}\right| |z - z_0|^n \leq 1$$

for $|z - z_0| \leq (1 - |z_0|)R$. To see this, we simply apply a linear change of variables, mapping the unit disk to the disk centered at z_0 of radius $1 - |z_0|$. This allows us to put the above criterion in invariant form.

Theorem 3.2. *(Invariant criterion) Let X be a Banach space of analytic functions on the disk as in the previous theorem satisfying condition (*). Then Bohr's*

4. Functions of several variables

The above scheme applies to a several variables context as well. We will use the standard multivariate notations as in [9]: we write an n-variable power series $\sum_\alpha c_\alpha z^\alpha$ where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is an n-tuple of non-negative integers, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, $\alpha! = \alpha_1!\alpha_2!...\alpha_n!$, $z = (z_1, z_2, ..., z_n)$ is an n-tuple of complex numbers, and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} ... z_n^{\alpha_n}$. We consider analytic functions defined on the unit polydisk $D^n = \{z : \max_{1\le j\le n} |z_j| < 1\}$. We will denote by D^α the derivative

$$\frac{\partial^{|\alpha|}}{(\partial z_1)^{\alpha_1} ... (\partial z_n)^{\alpha_n}}.$$

The following theorem is a several variable analogue of Theorem 2.5, b). (Extensions of most of the other results in Section 2 could be carried out in a manner similar to that of [11], and we omit them.)

Theorem 4.1. *Let* $0 < p < 2$ *and* $q = \frac{2}{2-p}$. *Let* $f(z) = \sum_\alpha c_\alpha z^\alpha \in H^2(D^n)$ *be such that*

$$\|f\|_{H^2(D^n)} := \sum_{|\alpha|=0}^{\infty} |c_\alpha|^2 \le 1$$

and $c_0 = 0$. *Then*

$$\sum_\alpha |c_\alpha|^p |z^\alpha| < 1$$

for any $z \in R_n D^n$, *where* $R_n \le (\frac{1}{2n})^{\frac{1}{q}}$.

Proof. Let $z = Rw$ for $w \in D^n$. Then

$$\begin{aligned}
\sum_\alpha |c_\alpha|^p |z^\alpha| &= \sum_\alpha |c_\alpha|^p |(Rw)^\alpha| \\
&= \sum_\alpha |c_\alpha|^p |w^\alpha| R^{|\alpha|} \\
&\le \Big(\sum_\alpha (|c_\alpha|^p |w^\alpha|)^{\frac{2}{p}}\Big)^{\frac{p}{2}} \Big(\sum_\alpha |R^{|\alpha| q}\Big)^{\frac{1}{q}} \\
&\qquad \text{(by Hölder's inequality)} \\
&= \Big(\sum_\alpha (|c_\alpha|^2 |w^\alpha|^{\frac{2}{p}})\Big)^{\frac{p}{2}} \Big(\sum_{k=1}^{\infty} \sum_{|\alpha|=k} R^{kq}\Big)^{\frac{1}{q}} \\
&\le \Big(\sum_\alpha |c_\alpha|^2\Big) \Big(\sum_{k=1}^{\infty} (R^q)^k n^k\Big)^{\frac{1}{q}} \\
&\le 1 \qquad \text{(when } R \le (\frac{1}{2n})^{\frac{1}{q}}\text{)}.
\end{aligned}$$

□

In a similar manner, we can extend Theorem 3.1 to several variables.

Theorem 4.2. *Let X be a Banach space of analytic functions from D^n into $\mathbb{C}$ such that polynomials are dense in X, the set of bounded point evaluations is D^n, and if $f \in X$ with $\|f\|_X \le 1$ is not constant then $|f(0)| < 1$. Then Bohr's phenomenon holds in X if and only if*

$$\sup\left\{\left(\frac{|(D^\alpha f)(0)|}{\alpha!(1-|f(0)|)}\right)^{\frac{1}{|\alpha|}} : \|f\|_X \le 1, |f(0)| < 1, \alpha \in \mathbb{N}^n, \alpha \ne 0\right\} < \infty.$$

We leave it to the reader to restate Theorem 4.2 in a point-invariant form similarly to Theorem 3.2.

In particular, if we are interested in bounded functions on the polydisk, we have a several variable analogue of Corollary 3.4.

Corollary 4.3. *Let f be an analytic function from D^n to D. Then for each multi-index α,*

$$\sup_{z\in D^n} \frac{|(D^\alpha f)(z)|(1-|z_1|^2)^{\alpha_1}(1-|z_2|^2)^{\alpha_2}...(1-|z_n|^2)^{\alpha_n}}{1-|f(z)|^2} < \infty. \tag{6}$$

Notice that applying (6) coordinate-wise, one can easily extend this result to mappings from the polydisk into itself. Moreover, since as is shown in [9], Bohr's radius for any complete Reinhardt domain G is positive, Corollary 4.3 immediately extends to all such domains and accordingly, for example to holomorphic mappings of the unit ball into itself, a recent result of MacCluer, Stroethoff, and Zhao (cf. [19].)

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